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Pattern-Specific Neural Network Design

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We present evidence that the performance of the traditional fully connected Hopfield model can be dramatically improved by carefully selecting an information-specific connectivity structure, while the synaptic weights of the selected connections are the same as in the Hopfield model. Starting from a completely disconnected network we let "genuine" Hebbian synaptic connections grow, one by one, until a desired degree of stability is achieved. Neural pathways are thus fixed not *before*, but *during* the learning phase.

KEY WORDS: Information storage; Hopfield model; specific connectivity.

INTRODUCTION

Successful performance of a particular neural network model with application potential depends crucially on the designer's ability to choose a suitable connectivity structure, the arrangement of nodes and patterns of connections, adapted to a context-dependent task. Though a variety of learning algorithms for optimal network design have been proposed in recent years,[1-3] many common models are based on *fixed* connectivity structures, necessarily far from being optimal. However, it seems conceivable that, particularly for large neuronal assemblies, *sparse* and highly specific connectivity structures have to be designed with great care and ingenuity. On one hand, a network model should have a *minimal* degree of connectivity; on the other, it should achieve a *maximal* retrieval quality. Hence, a delicate tradeoff between economical interconnectivity and high network performance emerges.

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1. STABILITY OF THE COMPLETELY CONNECTED NETWORK

Let us consider a set of N binary model units described by the state variables σ_i only capable to take the value $+1$ or -1 corresponding to states of "high" or "low" activity, respectively. The connectivity structure for each individual unit i is such that each neuron i receives input from K_i other units with $1 \leqslant K_i \leqslant N-1$ such that self-interactions are excluded. The dynamical time evolution of the system in discrete time steps is then given by the prescription

$$\sigma_i(t+1) = \operatorname{sgn} \sum_{l=1}^{K_i} c_{ij_l(i)}\, \sigma_{j_l(i)}(t), \qquad i = 1,..., N \tag{1.1}$$

In order to embed a set of $p = \alpha N$ prescribed N-dimensional *binary* patterns of activity $\{S^1,..., S^p\}$, taking the values $S_i^\mu = \pm 1$ with equal probability, as fixed points of the dynamics (1.1), Hopfield[1] suggested a completely connected network equipped with *symmetrical* couplings given by the Hebbian prescription

$$c_{ij}^{\mathrm{H}} = \sum_{\mu=1}^{p} S_i^\mu S_j^\mu \tag{1.2}$$

Note, however, that Hopfield's dynamical time evolution is stochastic, in contrast to the deterministic and synchronous dynamics defined in Eq. (1.1). Furthermore, due to incomplete connectivity the connectivity matrix $\mathbf{C} = (c_{ij})$ is in general no longer symmetric. Our minimal condition for successful information storage is that all $p \cdot N$ spherically normalized stability parameters $\kappa_{i\mu}$, defined by

$$\kappa_{i\mu} = S_i^\mu \sum_{l=1}^{K_i} c_{ij_l} S_{j_l}^\mu \Big/ \left(\sum_{l=1}^{K_i} c_{ij_l}^2 \right)^{1/2} \tag{1.3}$$

be larger than zero, while their individual magnitude can be considered as a measure for the embedding strength of the ith component of pattern μ. Since the probability distribution $\rho(\kappa)$ of the stability parameters $\kappa_{i\mu}$ is a Gaussian distribution centered at the mean value $\bar{\gamma} = 1/\sqrt{\alpha}$ and variance unity, the probability of finding a stability parameter *larger* than a prescribed value κ is obtained by integrating the probability distribution $\rho(\gamma)$,

$$P(\kappa_{i\mu} > \kappa) = \int_{\kappa}^{\infty} \rho(\gamma)\, d\gamma = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\alpha}} - \kappa \right) \right) \right] \tag{1.4}$$

with the usual definition of the error function $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x \exp(-t^2)\, dt$. The probability that for fixed i *all* p of the $\kappa_{i\mu}$ be larger than κ is then

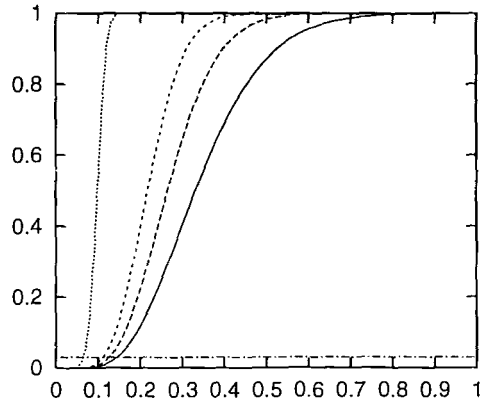

Fig. 1. Theoretic $P_{\min}(N, \alpha, 0)$ as a function of α for $N = 50$, 100, 200, and 10,000 (from right to left).

$[P(\kappa_{i\mu} > \kappa)]^p$. Hence, the probability of finding at least *one* $\kappa_{i\mu}$ *smaller* than or equal to κ is given by

$$P_{\min}(N, \alpha, \kappa) = 1 - [P(\kappa_{i\mu} > \kappa]^{\alpha N} \tag{1.5}$$

For the crucial case $\kappa = 0$, $P_{\min}(N, \alpha, 0)$ gives the fraction of not successfully embedded patterns with respect to component i. Figure 1 depicts this quantity for various numbers N in the Hopfield case.

The fraction of negative minimal stability parameters $\kappa_{i\mu}$ given by Eq. (1.5) increases monotonically with increasing load parameter α. Note, however, that even for small values of α, $P_{\min}(N, \alpha, 0)$ never takes the value zero such that there is always a finite chance that a microscopic fraction of patterns violates the embedding condition for fixed i. Hence we tolerate a fraction of negative minimal stability parameters not exceeding 3% and find critical values $\alpha_c^H(50) = 0.14$, $\alpha_c^H(100) = 0.13$, $\alpha_c^H(200) = 0.11$, and $\alpha_c^H(10{,}000) = 0.065$. Thus, with increasing N the critical $\alpha_c^H(N)$, i.e., the onset of macroscopic instability, shifts to smaller values of α, while $P_{\min}(N, \alpha, 0)$ eventually approaches a step function, taking only the value 0 for $\alpha = 0$ and the value 1 otherwise.

2. PARTIAL CONNECTIVITY STRUCTURES

We now focus on improving the performance of the fully connected Hopfield network model measured by the fraction of negative minimal stability parameters as given in Eq. (1.5) and visualized in Fig. 1. Our

intention is to reduce the number of connections, while the values of the surviving weights are still specified by Eq. (1.2). Thus learning can be viewed as adapting the network *architecture* to a context-dependent task rather than fine tuning the values of the *couplings*. In fact, for sparsely connected networks it is conceivable that the choice of an appropriate connectivity structure is of high priority. In this spirit we start from a completely disconnected network and let the number of connections grow until a desired value of the cost function

$$E(K_i, j_1(i), j_2(i),..., j_{K_i}(i)) = \min_{\mu=1}^{p}(\kappa_{i\mu}) = \min_{\mu=1}^{p}\left(S_i^\mu \sum_{l=1}^{K_i} c_{ij_l} S_{j_l}^\mu\right) \tag{2.1}$$

is achieved.[4, 5] Alternatively one could also maximize (2.1). Note that the choice of the cost function (2.1) consistent with Eq. (1.5) assigns a measure for the embedding strength of the "weakest" pattern with respect to component i.

Assuming that the units are arranged in arbitrary, but fixed order, a general recipe for the construction of the connectivity structure of unit i is as follows:

1. Start with an arbitrary unit labeled $j_1(i)$ that serves as the first neuron feeding input to unit i;

2. If unit i is already partially connected to s other units labeled as $j_1(i),..., j_s(i)$ $(s < N-1)$, the first unit of the remaining $N-1-s$ "trial" neurons that successfully fulfills the stability test

$$E(s+1, j_1(i), j_2(i),..., j_s(i), j_{\text{trial}}(i)) > E(s, j_1(i), j_2(i),..., j_s(i)) \tag{2.2}$$

joins the current members of the partially connected network. If all trial units fail, accept an arbitrary unit of the current trial set.

3. Stop when the cost function (2.1) exceeds or equals a prescribed stability value $\kappa > 0$. Otherwise go back to step 2.

Since process 1–3 has to be performed for each unit i the construction of a partially connected network where K_i units are connected to unit i demands at most $\sum_{i=1}^{N}(N-1-K_i/2)\times(K_i-1)$ comparisons (2.2).

3. COMPUTER SIMULATIONS

Evidently the network structure resulting from our deterministic recipe may strongly depend on the sequential order of the units. Hence we study the dependence on random order (strategy A) and the order according to

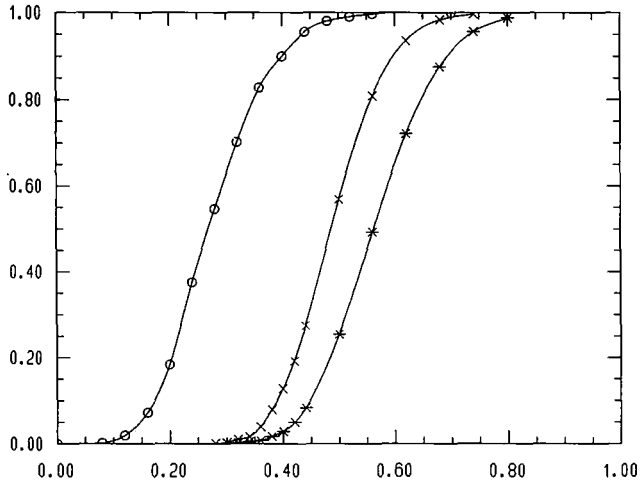

Fig. 2. Fraction of negative minimal stabilities as a function of α for a fully connected network (coincides with theory), selective connectivity according to strategy A and selective connectivity according to strategy B (from right to left).

the magnitude of the Hebbian couplings (strategy B). Figure 2 depicts the overall performance of our selectively connected networks for both learning strategies and $N=100$ neurons. For comparison we add the theoretical prediction (1.5) for the completely connected network which coincides with our computer experiments. Tolerating a total error of 3% percent, we find the remarkable feature that the number of patterns which can be successfully embedded is about three times lower for the traditional fully connected Hopfield model $\alpha_c^{\mathrm{H}}(100)=0.13$ than for the selectively constructed network models. To be more precise, the critical values of α shift to $\alpha_c^{\mathrm{A}}(100)=0.37$ and $\alpha_c^{\mathrm{B}}(100)=0.39$ for strategy A and strategy B, respectively. We note that the shift of the critical load parameters $\alpha_c^{\mathrm{H}}(N)-\alpha_c^{\mathrm{B}}(N)$ does not show a sensitive dependence on N up to $N=200$, though we have to expect deviations for larger values of N, since our problem is presumably *NP*-complete.

During the following computer experiments we fix the number of units N as well as the load parameter α and stop our procedure when the cost function (2.1) attains it maximal value. For $N=100$ and $\alpha=0.3$ we find average minimal stability values $\langle\kappa^{\mathrm{A}}\rangle=0.557$ and $\langle\kappa^{\mathrm{B}}\rangle=0.726$ for strategy A and strategy B, respectively. The distribution of the *selected* couplings for both strategies is shown in Fig 3. For comparison we add the nearly binomial distribution of the full set of Hebbian couplings *before* the selection process.

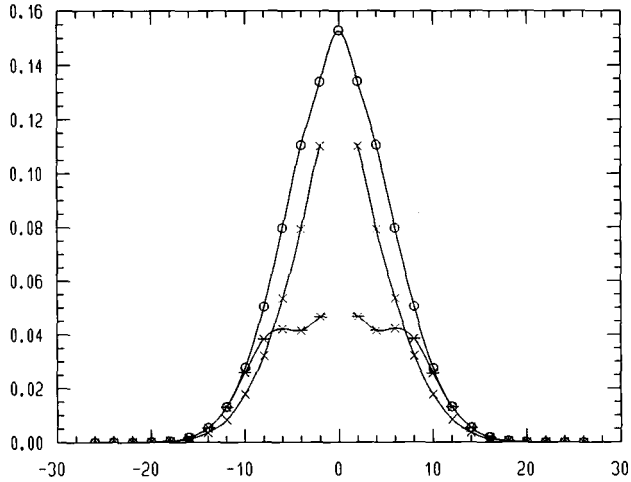

Fig. 3. Distribution of the weights *before* (circles) and *after* the selection process for $N = 100$, load parameter $\alpha = 0.30$, and fully optimized κ for strategy A (crosses) and strategy B (stars).

For strategy B we observe some fluctuations (structure) in the region of low weights due to the special rearrangement of the candidates. By contrast, strategy A based on random order reveals a close to binomial shape similar to the completely connected model. Note that the fraction of zero couplings increases from 0.153 to 0.386 and 0.568 after application of

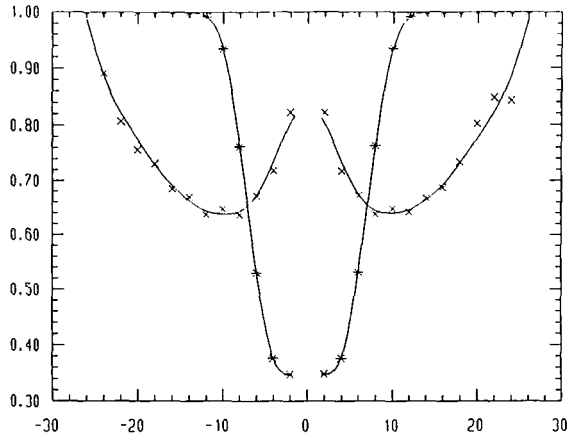

Fig. 4. Acceptance rate distribution for the Hebbian weights for strategy A (crosses) and strategy B (stars).

strategy A and strategy B, respectively. Hence, the superiority of strategy B is twofold, leading to higher dilution as well as to better stability.

The distribution of the acceptance rate of the Hebbian couplings is shown in Fig. 4. While strategy B evidently favors the acceptance of almost all high weights, the acceptance rate for the smaller weights decreases almost linearly with the magnitude of the Hebbian weights. By contrast, the acceptance rate for strategy A can be approximated by a parabola favoring the high as well as very low weights.

4. OUTLOOK

We have seen that network architectures can be adapted to a specific pattern set with the aid of a simple deterministic step-by-step strategy such that with half as many couplings we could store three times more patterns.

Evidently, our search strategy for optimal network structures can be greatly improved. To this end genetic algorithms may be fruitful. A family of genotypes can naturally be specified by all $(N-1)!$ permutations of the lexicographic arrangement of the potential connectivity candidates. A deterministic procedure for the construction of the corresponding phenotypes as well as the introduction of a fitness function that measures how well a phenotype performs have been presented in this work.

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